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THE CIRCULAR DISK STRADDLING THE INTERFACE OF A TWO-PHASE FLOW

K. B. Ranger
Department of Mathematics, University of Toronto, Toronto, Canada M5S1A1

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Abstract—The motion is determined for a thin circular disk straddling the plane interface of an immiscible two phase creeping flow and moving parallel or perpendicular to the interface. Expressions are derived for the drag coefficient on the disk.

INTRODUCTION

The motion of a particle moving in the presence of a free fluid–fluid interface is of some importance and interest in chemical engineering science. The general motion of an arbitrary particle in the presence of such a free interface is a matter of considerable complexity, and Chadwick (1976) in some unpublished notes has considered the particular cases of a concentrated point force and a sphere in a two-phase flow.

In this paper the simplest geometry of particle shape is investigated, namely, the thin circular disk and in all cases the disk straddles the interface which is assumed to be instantaneously planar. The first flow described is the situation of a disk straddling an interface and moving parallel to it. The flow is asymmetric and a new representation of the solution of the creeping flow equations is given to construct the velocity field. The velocity and stress are continuous at the interface and this leads to mixed boundary value problems for the Stokes equation. It is found that the velocity is independent of the ratio of viscosities of the two phases and the flow is the same as if the disk were moving edgewise on through an infinite fluid of the same viscosity. This follows from the symmetry of the disk where the normal and tangential components of stress vanish identically on the interface. The drag coefficient on the disk is a simple modification of the drag given for a disk by Oberbeck (1945). The representation for the velocity field is new and is obtained using the method of complementary integral representations described by Ranger (1972).

Section 2 generalizes the problem to an axisymmetric body and here only an approximate solution is presented. It is shown that the axisymmetric Stokes flow past the body satisfies the conditions of continuous velocity and tangential stress at the interface and only the condition of continuous normal stress is not satisfied. The case of a sphere is considered in detail and it is shown the difference in normal stress is proportional to the difference in viscosities and proportional to the inverse fourth power of distance measured from the sphere center. The solution is thus a good approximation to the exact solution at large distances and the drag in the two phase flow is related in a simple manner to that in ordinary Stokes flow.

Section 3 deals with the situation in which the disk straddles the interface and is instantaneously moving perpendicular to it. In this model the velocity and stress are continuous at the interface and it is found that the velocity field is independent of the ratio of viscosities and the field is the same as if the disk were moving broadside on in an infinite fluid of the same viscosity. The drag on the disk is again related in a simple way to the drag in ordinary Stokes flow. It is worth pointing out that the disk is the only geometry for which an explicit exact solution can be determined for the velocity field. The case of the sphere is excessively complicated and it is not clear that a solution for the velocity field exists with continuous velocity and stress at the interface.

Section 4 again deals with the disk in the interface and moving perpendicular to it. However, here it is assumed the normal velocity on the interface is zero and that the interfacial tension is sufficiently high to preclude large deformations of the interface. The velocity is thus discontinuous at the rim of the disk and this leads to an infinite drag on the disk. A solution for the velocity field does exist and is uniquely determined using the principle of minimum singularity. It is found that the velocity fluid is again independent of the viscosity ratio and the velocity vanishes on the interface. The flow thus behaves like a circular disk moving through a plane containing a circular gap. There appears to be no analogue of this flow for an axisymmetric body with non zero volume.

1. EQUATION OF MOTION AND DISK MOVING PARALLEL TO INTERFACE

The equations of steady creeping flow are

$$\operatorname{grad} p = \mu \nabla^2 \mathbf{q}, \qquad \operatorname{div} \mathbf{q} = 0, \tag{1}$$

where $\mathbf{q}$ is the fluid velocity, p the pressure and μ the viscosity. A suitable representation for the fluid velocity field in which boundary conditions are prescribed on the plane $x = 0$, is expressed by

$$\mathbf{q} = \operatorname{curl}^2 \left\{ \frac{\psi}{\rho}\, \hat{i} \cos\phi \right\} + \operatorname{curl} \left\{ \frac{\chi}{\rho}\, \hat{i} \sin\phi \right\}, \tag{2}$$

where (x, ρ, ϕ) denote cylindrical polar coordinates and the scalar functions ψ and χ satisfy

$$L_{-1}^2(\psi) = 0, \qquad L_{-1}(\chi) = 0 \tag{3}$$

and the Stokes operator L_{-1} is defined by

$$L_{-1} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}. \tag{4}$$

Consider now a two-phase flow in which the fluid in the region $x \geq 0$ has viscosity μ_1 and in the region $x \leq 0$ the viscosity is μ_2. A circular thin disk defined by $x = 0$, $0 \leq \rho \leq 1$ straddles and moves in the interface $x = 0$ with velocity given by

$$\mathbf{q}_0 = \hat{\rho} \cos\phi - \hat{\phi} \sin\phi. \tag{5}$$

The fluid velocity in the two phases can be represented by

$$\mathbf{q}_j = \operatorname{curl}^2 \left\{ \frac{\psi_j}{\rho}\, \hat{i} \cos\phi \right\} + \operatorname{curl} \left\{ \frac{\chi_j}{\rho}\, \hat{i} \sin\phi \right\} \tag{6}$$

where ψ_j and χ_j satisfy [3] and $j = 1, 2$. If $\mathbf{q}_j = u_x^{(j)} + u_\rho^{(j)}\hat{\rho} + u_\phi^{(j)}\hat{\phi}$, then

$$u_x^{(j)} = \left\{ \frac{1}{\rho}\frac{\partial^2 \psi_j}{\partial x^2} - \frac{1}{\rho} L_{-1}(\psi_j) \right\} \cos\phi, \tag{7}$$

$$u_\rho^{(j)} = \left\{ \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi_j}{\partial x} \right) + \frac{\chi_j}{\rho^2} \right\} \cos\phi, \tag{8}$$

$$u_\phi^{(j)} = -\left\{ \frac{1}{\rho^2}\frac{\partial \psi_j}{\partial x} + \frac{\partial}{\partial \rho}\left(\frac{\chi_j}{\rho} \right) \right\} \sin\phi. \tag{9}$$

An appropriate representation for ψ_j is

$$\psi_j = xV_j, \qquad L_{-1}(V_j) = 0, \qquad j = 1, 2. \tag{10}$$

Now on the disk $x = 0$, $0 \le \rho \le 1$

$$u_x^{(j)} = 0, \qquad u_\rho^{(j)} = \cos\phi, \qquad u_\phi^{(j)} = -\sin\phi, \tag{11}$$

and in terms of V_j and χ_j, [11] are equivalent to

$$V_j = \frac{1}{2}\rho^2, \qquad \chi_j = \frac{1}{2}\rho^2 \qquad \text{at } x = 0, \qquad 0 \le \rho \le 1. \tag{12}$$

The normal component of velocity

$$u_x^{(j)} = 0, \qquad x = 0, \qquad \rho = 1 \tag{13}$$

so that the condition of zero normal velocity on the interface is satisfied as well as on the disk. The boundary conditions on the interface require continuity of the velocity as well as stress on $x = 0$, $\rho > 1$. These will be treated in turn.

Continuity of tangential components of velocity on the interface

These conditions are equivalent to

$$\left.\begin{aligned} \frac{\partial}{\partial\rho}\left(\frac{V_1 - V_2}{\rho}\right) + \frac{\chi_1 - \chi_2}{\rho^2} = 0, \\ \frac{1}{\rho^2}(V_1 - V_2) + \frac{\partial}{\partial\rho}\left(\frac{\chi_1 - \chi_2}{\rho}\right) = 0, \end{aligned}\right\} \tag{14}$$

at $x = 0$, $\rho > 1$. These boundary conditions can be satisfied by taking

$$V_1 = V_2, \qquad \chi_1 = \chi_2 \qquad \text{at } x = 0, \rho > 1. \tag{15}$$

Normal component of stress continuous at interface

The normal component of stress is expressed by

$$p_{xx}^{(j)} = -p_j + 2\mu_j \frac{\partial u_x^{(j)}}{\partial x} \tag{16}$$

where the pressure p_j is found from [6] and is given by

$$p_j = \frac{\mu_j}{\rho}\frac{\partial}{\partial x} L_{-1}(\psi_j). \tag{17}$$

In terms of the stream function ψ_j [16] becomes

$$\begin{aligned} p_{xx}^{(j)} &= \left\{\frac{-\mu_j}{\rho}\frac{\partial}{\partial x}L_{-1}(\psi_j) + \frac{2\mu_j}{\rho}\left[\frac{\partial^2\psi_j}{\partial x^2} - \frac{\partial}{\partial x}L_{-1}(\psi_j)\right]\right\}\cos\phi \\ &= \left\{\frac{2\mu_j}{\rho}\frac{\partial^3\psi_j}{\partial x^3} - \frac{3\mu_j}{\rho}\frac{\partial}{\partial x}L_{-1}(\psi_j)\right\}\cos\phi \\ &= 0, \qquad x = 0, \rho > 1. \end{aligned} \tag{18}$$

Thus

$$p_{xx}^{(1)} = p_{xx}^{(2)} = 0 \qquad \text{on } x = 0,\ \rho > 1. \qquad [19]$$

Tangential components of stress continuous at interface

There are two components of tangential stress $p_{\rho x}^{(j)}$ and $p_{\phi x}^{(j)}$. The first is given by

$$p_{\rho x}^{(j)} = \mu_j\left\{\frac{\partial u_x^{(j)}}{\partial \rho} + \frac{\partial u_\rho^{(j)}}{\partial x}\right\} \qquad [20]$$

$$= \mu_j\left\{2\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial V_j}{\partial x}\right) + \frac{1}{\rho^2}\frac{\partial \chi_j}{\partial x}\right\}\cos\phi. \qquad [21]$$

The stress [21] is continuous at the interface if

$$p_{\rho x}^{(1)} = p_{\rho x}^{(2)} \qquad \text{at } x = 0,\ \rho > 1 \qquad [22]$$

that is if

$$2\frac{\partial}{\partial \rho}\left\{\frac{1}{\rho}\frac{\partial}{\partial x}(\mu_1 V_1 - \mu_2 V_2)\right\} + \frac{1}{\rho^2}\left(\mu_1\frac{\partial \chi_1}{\partial x} - \mu_2\frac{\partial \chi_2}{\partial x}\right) = 0, \qquad [23]$$

at $x = 0$, $\rho > 1$. The second tangential stress is given by

$$p_{\phi x}^{(j)} = \mu_j \frac{1}{\rho}\frac{\partial u_x^{(j)}}{\partial \phi} + \frac{\partial u_\phi^{(j)}}{\partial X} \qquad [24]$$

$$= \mu_j\left\{-\frac{1}{\rho^2}\frac{\partial^2 \psi_j}{\partial x^2} + \frac{1}{\rho^2}L_{-1}(\psi_j) - \frac{1}{\rho^2}\frac{\partial^2 \psi_j}{\partial x^2} - \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \chi_j}{\partial x}\right)\right\}\sin\phi \qquad [25]$$

$$= -\mu_j\left\{\frac{2}{\rho^2}\frac{\partial V_j}{\partial x} + \frac{\partial}{\partial \rho}\frac{1}{\rho}\frac{\partial \chi_j}{\partial x}\right\}\sin\phi. \qquad [26]$$

The stress [26] is continuous at the interface if

$$p_{\phi x}^{(1)} = p_{\phi x}^{(2)} \qquad \text{at } x = 0,\ \rho > 1. \qquad [27]$$

Equations [26] and [27] imply

$$\frac{1}{\rho^2}\frac{\partial}{\partial x}(\mu_1 V_1 - \mu_2 V_2) + \frac{\partial}{\partial \rho}\frac{1}{\rho}\frac{\partial}{\partial x}(\mu_1 X_1 - \mu_2 X_2). \qquad [28]$$

Equations [23] and [28] are both satisfied if

$$\mu_1\frac{\partial V_1}{\partial x} = \mu_2\frac{\partial V_2}{\partial x}, \qquad [29]$$

$$\mu_1\frac{\partial \chi_1}{\partial x} = \mu_3\frac{\partial \chi_2}{\partial x}$$

at $x = 0$, $\rho > 1$.

***Solution for* V_j *and* χ_j**

First consider the mixed boundary value problem for V_j. V_j satisfies

$$L_{-1}(V_j) = 0, \qquad [31]$$

subject to the conditions

$$V_j = \frac{1}{2}\rho^2, \qquad \text{at } x = 0,\ 0 \le \rho \le 1 \tag{32}$$

on the disk. On the interface

$$V_1 = V_2 \qquad \text{at } x = 0,\ \rho > 1 \tag{33}$$

$$\mu_1 \frac{\partial V_1}{\partial x} = \mu_2 \frac{\partial V_2}{\partial x} \qquad \text{at } x = 0,\ \rho > 1. \tag{34}$$

In addition the fluid velocity vanishes at infinity which is satisfied if $V_j \to 0$ as $x^2 + \rho^2 \to \infty$. The mixed boundary value problem for χ_j is the same as that posed by [31] to [34] with χ_j replacing V_j.

Method of complementary integral representations

Appropriate integral representations for $V_j(x, \rho)$, see Ranger (1972), are:

$$V_1(x, \rho) = \int_0^\rho \frac{v_1(x, y)y\, dy}{(\rho^2 - y^2)^{1/2}} = \int_\rho^\infty \frac{u_1(x, y)y\, dy}{(y^2 - \rho^2)^{1/2}} \tag{35}$$

for $x \ge 0$

$$V_2(x, \rho) = \int_0^\rho \frac{v_2(-x, y)\, dy}{(\rho^2 - y^2)^{1/2}} = \int_\rho^\infty \frac{u_2(-x, y)y\, dy}{(y^2 - \rho^2)^{1/2}} \tag{36}$$

for $x \le 0$. (u_j, v_j) are conjugate two dimensional harmonics even and odd in y respectively and expressible in the form

$$u_j + iv_j = \int_0^\infty f_j(k)\, e^{-k(x+iy)}\, dk \tag{37}$$

where $f_j(k)$ are real functions of k.

Now on the disk

$$V_j(o, \rho) = \frac{1}{2}\rho^2 = \int_0^\rho \frac{v_j(o, y)y\, dy}{(\rho^2 - y^2)^{1/2}}, \qquad 0 \le \rho < 1. \tag{38}$$

The inverse of the Abel type integral equation is

$$v_j(o, y) = \frac{2}{\pi y}\frac{\partial}{\partial y}\int_0^y \frac{V_j(o, \rho)\rho\, d\rho}{(y^2 - \rho^2)^{1/2}} = \frac{2y}{\pi}. \qquad 0 \le y < 1 \tag{39}$$

and $j = (1, 2)$. Again on the interface

$$V_1(o, \rho) - V_2(o, \rho) = \int_0^\rho \frac{[v_1(o, y) - v_2(o, y)]y\, dy}{(\rho^2 - y^2)^{1/2}} \tag{40}$$

$$= \int_0^\rho \frac{[v_1(o, y) - v_2(o, y)]y\, dy}{(\rho^2 - y^2)^{1/2}} = 0 \tag{41}$$

for $\rho > 1$. Hence from [41]

$$v_1(o, y) = v_2(o, y) \qquad \text{for } |y| > 1. \tag{42}$$

Also on the interface

$$\left(\mu_1 \frac{\partial V_1}{\partial x} - \mu_2 \frac{\partial V_2}{\partial x}\right)\bigg|_{x=0} = \int_\rho^\infty \frac{\left(\mu_1 \frac{\partial u_1}{\partial x} + \mu_2 \frac{\partial u_2}{\partial x}\right)\big|_{x=0} y\, \mathrm{d}y}{(y^2 - \rho^2)^{1/2}} = 0 \qquad [43]$$

for $\rho > 1$, and this implies

$$\mu_1 \frac{\partial u_1}{\partial x} + \mu_2 \frac{\partial u_2}{\partial x} = \mu_1 \frac{\partial v_1}{\partial y} + \mu_2 \frac{\partial v_2}{\partial y} = 0, \qquad x = 0, |y| > 1 \qquad [44]$$

or equivalently

$$\mu_1 v_1(o, y) + \mu_2 v_2(o, y) = A, \qquad |y| > 1 \qquad [45]$$

where A is a constant. Thus,

$$v_1(o, y) = v_2(o, y) = \frac{A}{\mu_1 + \mu_2} = B \text{ say.} \qquad [46]$$

for $|y| > 1$. It is observed that

$$\mu_1 \frac{\partial V_1}{\partial x} = \mu_2 \frac{\partial V_2}{\partial x} = 0, \qquad \text{for } x = 0, \rho > 1. \qquad [47]$$

so that the stress vanishes over the interface since a similar result is valid for χ_1 and χ_2.

The solution for $v_j(x, y)$ can now be determined by standard Green's function methods and is expressed by

$$v_j(x, y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{v_j(o, s)\, \mathrm{d}s}{x^2 + (s - y)^2} \qquad [48]$$

where

$$\left.\begin{aligned} v_j(o, s) &= \frac{2}{\pi} s \qquad && |s| < 1 \\ &= B && s > 1 \\ &= -B && s < -1. \end{aligned}\right\} \qquad [49]$$

Thus

$$V_j(x, \rho) = X_j(x, \rho) = \int_0^\rho \frac{v_j(x, y) y\, \mathrm{d}y}{(\rho^2 - y^2)^{1/2}}, \qquad x > 0 \qquad [50]$$

where $v_j(x, y)$ is determined from [48] and [49]. Explicit evaluation of [48] yields

$$v_j(x, y) = \frac{2y}{\pi^2}\left\{\tan^{-1} \frac{1-y}{x}\right\} - \tan^{-1}\left\{\frac{-1-y}{x}\right\} + \frac{x}{\pi^2} \log\left[\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}\right]$$
$$+ \frac{B}{\pi}\left[\tan^{-1}\left(\frac{-1-y}{x}\right) - \tan^{-1}\left(\frac{1-y}{x}\right)\right]. \qquad [51]$$

To determine the constant B, consider the velocity components on $x = 0$, $\rho > 1$. It suffices to consider $u_\rho^{(j)}$ which is given by

$$u_\rho^{(j)} = \left\{\frac{\partial}{\partial \rho}\left(\frac{1}{\rho} v_j(o, \rho)\right) + \frac{V_j(o, \rho)}{\rho^2}\right\} \cos\phi \qquad [52]$$

where on calculation

$$V_j(o,\rho)=\frac{\rho^2}{\pi}\sin^{-1}\left(\frac{1}{\rho}\right)+\left(B+\frac{1}{\pi}\right)(\rho^2-1)^{1/2}. \quad [53]$$

Hence,

$$u_\rho^{(j)}=\frac{1}{\rho}\frac{\partial V_j}{\partial \rho}(o,\rho)\cos\phi \quad [54]$$

$$=\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\rho}\right)-\frac{1}{\pi(\rho^2-1)^{1/2}}+\left(B-\frac{1}{\pi}\right)\frac{1}{(\rho^2-1)^{1/2}}\cos\phi. \quad [55]$$

Since the velocity is finite as $\rho\to 1+$, the singularity is eliminated by choosing $B=2/\pi$. $v_j(x,y)$ is now given by

$$v_j(x,y)=\frac{2}{\pi^2}(y-1)\left\{\tan^{-1}\left(\frac{1-y}{x}\right)-\tan^{-1}\left(\frac{-1-y}{x}\right)\right\}+\frac{x}{\pi^2}\log\frac{x^2+(1-y)^2}{x^2+(1+y)^2}. \quad [56]$$

Substitution of [56] in [50] yields $V_j(x,\rho)=\chi_j(x,\rho)$. Now the velocity field is clearly independent of the ratio of the viscosities of the two phases so that basically the solution represents the edgewise motion of a disk through an infinite fluid. This follows from the fact that the stress vanishes over the interface. The drag on a disk of radius a moving with speed V is then

$$D=\frac{16}{3}Va(\mu_1+\mu_2). \quad [57]$$

2. APPROXIMATE SOLUTION FOR AXISYMMETRIC BODIES MOVING PARALLEL TO AN INTERFACE

Again, let (x,ρ,ϕ) be cylindrical polar coordinates and consider an axisymmetric body S with the x axis as axis of symmetry moving parallel to an interface $\phi=\pm\pi/2$ with velocity

$$\mathbf{q}_0=V\hat{i} \quad [58]$$

in an infinite two phase flow. The viscosity for $-\pi/2\le\phi\le\pi/2$ is μ_1 and for $\pi/2\le\phi\le\pi$, $-\pi/2\ge\phi\ge-\pi$ is μ_2. The flow is clearly asymmetric but an approximate axisymmetric motion will be considered.

Let the complete flow field be represented by

$$\mathbf{q}=\operatorname{curl}\left(\frac{-\psi}{\rho}\hat{\phi}\right) \quad [59]$$

where

$$\mathbf{q}=u_x\hat{i}+u_\rho\hat{\rho}$$

and

$$u_x=-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho},\qquad u_\rho=\frac{1}{\rho}\frac{\partial\psi}{\partial x}. \quad [60]$$

ψ satisfies the Stokes repeated operator equation

$$L_{-1}^2(\psi)=0 \quad [61]$$

where L_{-1} is defined by [4]. On the boundary of S

$$-V=\frac{1}{\rho}\frac{\partial\psi}{\partial\rho},\qquad 0=\frac{1}{\rho}\frac{\partial\psi}{\partial x}. \tag{62}$$

At infinity $\psi\sim 0(r)$, $r=(\rho^2+x^2)^{1/2}\sim\infty$, that is, the flow behaves like a Stokes couplet located at the origin. The velocity defined by [59] is continuous at the interface $\phi=\pm\pi/2$. The components of tangential stress on the interface are

$$p_{\rho\phi}=\frac{1}{2\rho}\frac{\partial}{\partial\phi}u_\rho=0. \tag{63}$$

and

$$p_{x\phi}=\frac{1}{2\rho}\frac{\partial u_x}{\partial\phi}=0. \tag{64}$$

Hence, the velocity and tangential stress are continuous at the interface and the only condition not satisfied is the continuity of normal stress. This is expressed by

$$p^{(1)}_{\phi\phi}=p^{(2)}_{\phi\phi}, \tag{65}$$

that is,

$$-p_1+\frac{2\mu_1}{\rho}u_\rho^{(1)}=-p_2+\frac{2\mu_2}{\rho}u_\rho^{(2)} \tag{66}$$

where the pressure p_j is obtained from the equations

$$\frac{\partial p_j}{\partial x}=-\frac{\mu_j}{\rho}\frac{\partial}{\partial\rho}L_{-1}(\psi),\qquad \frac{\partial p_j}{\partial\rho}=\frac{\mu_j}{\rho}\frac{\partial}{\partial x}L_{-1}(\psi). \tag{67}$$

If $x=r\cos\theta$, $\rho=r\sin\theta$, consider the particular case of a sphere expressed by $r=1$. The stream function is

$$\psi=\frac{V}{4}\left(3r-\frac{1}{r}\right)\sin^2\theta \tag{68}$$

and the pressure

$$p_j=-\frac{3V\mu_j\cos\theta}{2r^2}.$$

The normal stress is

$$p^{(j)}_{\phi\phi}=\frac{3V\mu_j\cos\theta}{2r^4}. \tag{69}$$

Thus, the difference in normal stress is

$$p^{(1)}_{\phi\phi}-p^{(2)}_{\phi\phi}=\frac{3V(\mu_1-\mu_2)\cos\theta}{2r^4} \tag{70}$$

which is small if either $(\mu_1-\mu_2)$ is small or if r is large. The drag on the sphere is

$$D=3\pi Va(\mu_1+\mu_2), \tag{71}$$

where the physical radius of the sphere is a. There is no component of force on the sphere perpendicular to the interface.

More generally if the drag in ordinary Stokes flow is known to be $6\pi Va\mu\alpha$, then the drag in two phase flow is given by

$$D = 3\pi Va(\mu_1 + \mu_2)\alpha. \tag{72}$$

3. AXISYMMETRIC FLOW, DISK MOVING PERPENDICULAR TO INTERFACE

For axisymmetric flow the velocity may be expressed by

$$\mathbf{q} = \operatorname{curl}\left\{\frac{-\psi}{\rho}\hat{\phi}\right\} \tag{73}$$

where ψ satisfies the Stokes repeated operation equation

$$L_{-1}^2(\psi) = 0 \tag{74}$$

and the operator L_{-1} is defined by [4]. The flow components are given by

$$u_x = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}, \qquad u_\rho = \frac{1}{\rho}\frac{\partial\psi}{\partial x}. \tag{75}$$

In the flow to be considered in this section the thin circular disk $x = 0$, $0 \le \rho \le 1$ straddles the interface $x = 0$ and is moving perpendicular to it. With the notation of the previous section, the stream functions for the flow in the two phases may be represented by ψ_j $(j = 1, 2)$. The velocity is then given by

$$\mathbf{q}_j = \operatorname{curl}\left\{\frac{-\psi_j}{\rho}\hat{\phi}\right\}, \qquad (j = 1, 2). \tag{76}$$

An appropriate representation for ψ_j is given by

$$\psi_j = U_j - x\left(\frac{\partial U_j}{\partial x} - V_j\right) \tag{77}$$

where V_j and U_j both satisfy

$$L_{-1}(U_j) = L_{-1}(V_j) = 0. \tag{78}$$

The boundary conditions on the disk require

$$\left.\begin{aligned} u_x^{(j)} &= -\frac{1}{\rho}\frac{\partial\psi_j}{\partial\rho} = -1 \\ u_\rho^{(j)} &= \frac{1}{\rho}\frac{\partial\psi_j}{\partial x} = 0 \end{aligned}\right\} \quad \text{at } x = 0,\ 0 \le \rho \le 1 \tag{79}$$

and these conditions are satisfied if

$$U_j = \frac{1}{2}\rho^2, \qquad V_j = 0, \qquad \text{at } x = 0,\ 0 \le \rho \le 1. \tag{80}$$

The velocity is continuous on the interface and this is satisfied providing

$$U_1 = U_2 \qquad \text{and} \qquad V_1 = V_2 \qquad \text{at } x = 0,\ \rho > 1. \tag{81}$$

Now the tangential component of stress is defined by

$$\begin{aligned} p_{\rho x}^{(j)} &= \mu_j\left(\frac{\partial u_x^{(j)}}{\partial \rho} + \frac{\partial u_\rho^{(j)}}{\partial x}\right) \\ &= \frac{\mu_j}{\rho}\left(\frac{\partial^2 \psi_1}{\partial x^2} - \rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi_1}{\partial \rho}\right)\right). \end{aligned} \tag{82}$$

The tangential component of stress is continuous at the interface, that is

$$p_{\rho x}^{(1)} = p_{\rho x}^{(2)} \qquad \text{at } x = 0,\ \rho > 1 \tag{83}$$

which reduces to

$$\mu_1\frac{\partial V_1}{\partial x} = \mu_2\frac{\partial V_2}{\partial x} \qquad \text{at } x = 0,\ \rho > 1. \tag{84}$$

The normal component of stress is defined by

$$p_{xx}^{(j)} = -p_j + 2\mu_j\frac{\partial u_x^{(j)}}{\partial x} = -p_j - \frac{2\mu_j}{\rho}\frac{\partial^2 \psi_j}{\partial x \partial \rho} \tag{85}$$

where p_j is the pressure obtained from the creeping flow equations

$$\frac{\partial p_j}{\partial \rho} = \frac{\mu_j}{\rho}\frac{\partial}{\partial x}L_{-1}(\psi_j), \qquad \frac{\partial p_j}{\partial x} = -\frac{\mu_j}{\rho}\frac{\partial}{\partial \rho}L_{-1}(\psi_j). \tag{86}$$

The normal stress is continuous at the interface

$$p_{xx}^{(1)} = p_{xx}^{(2)} \qquad \text{at } x = 0,\ \rho > 1 \tag{87}$$

which is equivalent to

$$\frac{\partial}{\partial \rho}p_{xx}^{(1)} = \frac{\partial}{\partial \rho}p_{xx}^{(2)} \qquad \text{at } x = 0,\ \rho > 1. \tag{88}$$

After some calculation it is found that [88] reduces to

$$\mu_1\frac{\partial^3 U_1}{\partial x^3} = \mu_2\frac{\partial^3 U_2}{\partial x^3} \qquad \text{at } x = 0,\ \rho > 1 \tag{89}$$

which is equivalent to

$$\mu_1\frac{\partial U_1}{\partial x} = \mu_2\frac{\partial U_2}{\partial X} \qquad \text{at } x = 0,\ \rho > 1. \tag{90}$$

In addition the velocity vanishes at infinity and this is satisfied if both

$$(U_j, V_j) \to 0 \qquad \text{as} \qquad x^2 + \rho^2 \to \infty. \tag{91}$$

To sum up the problems for U_j and V_j are mixed boundary value problems which may be stated in the form:

U_j satisfies [78] subject to the conditions

$$\begin{aligned} U_j &= \frac{1}{2}\rho^2 \qquad \text{at } x=0,\ 0\le\rho\le 1,\\ U_1 &= U_2 \qquad \text{at } x=0,\ \rho>1,\\ \mu_1\frac{\partial U_1}{\partial x} &= \mu_2\frac{\partial U_2}{\partial x} \qquad \text{at } x=0,\ \rho>1,\\ U_j &\to 0 \qquad \text{as} \qquad x^2+\rho^2\to\infty. \end{aligned} \tag{92}$$

The mixed problem for V_j is defined by

$$\begin{aligned} V_j &= 0, \qquad \text{at } x=0,\ 0\le\rho\le 1,\\ V_1 &= V_2, \qquad \text{at } x=0,\ \rho>1,\\ \mu_1\frac{\partial V_1}{\partial x} &= \mu_2\frac{\partial V_2}{\partial x} \qquad \text{at } x=0,\ \rho>1,\\ V_j &\to 0 \qquad \text{as} \qquad x^2+\rho^2\to\infty. \end{aligned} \tag{93}$$

It is readily shown the solution of the latter problem for V_j is identically zero.

To solve the problem for U_j ($j=1,2$) it is convenient as in the previous section to introduce complementary integral representations as follows:

$$U_j(x,\rho) = \int_0^\rho \frac{v_j(x,y)y\,\mathrm{d}y}{(\rho^2-y^2)^{1/2}} = \int_\rho^\infty \frac{u_j(x,y)y\,\mathrm{d}y}{(y^2-\rho^2)^{1/2}} \qquad x\ge 0 \tag{94}$$

$$= \int_0^\rho \frac{v_j(-x,y)y\,\mathrm{d}y}{(\rho^2-y^2)^{1/2}} = \int_\rho^\infty \frac{u_j(-x,y)y\,\mathrm{d}y}{(y^2-\rho^2)^{1/2}} \qquad x\le 0 \tag{95}$$

where u_j, v_j are conjugate two dimensional harmonic functions of x and y, even and odd in y respectively. The boundary conditions on the disk require

$$U_j(o,\rho) = \frac{1}{2}\rho^2 = \int_0^\rho \frac{y_j(o,y)y\,\mathrm{d}y}{(\rho^2-y^2)^{1/2}} \qquad \text{at } x=0,\ 0\le\rho\le 1. \tag{96}$$

The solution of the Abel type integral equation is

$$v_j(o,y) = \frac{2}{\pi}y, \qquad 0\le y\le 1. \tag{97}$$

On the interface

$$U_1(o,\rho) - U_2(o,\rho) = \int_0^\rho \frac{[v_1(o,y)-v_2(o,y)]y\,\mathrm{d}y}{(\rho^2-y^2)^{1/2}} = \int_1^\rho \frac{[v_1(o,y)-v_2(o,y)]y\,\mathrm{d}y}{(\rho^2-y^2)^{1/2}} = 0,$$

so that

$$v_1(o,y) = v_2(o,y), \qquad y>1. \tag{98}$$

Again,

$$\left(\mu_1\frac{\partial V_1}{\partial x} - \mu_2\frac{\partial V_2}{\partial x}\right)\Bigg|_{x=0} = \int_\rho^\infty \left(\mu_1\frac{\partial u_1}{\partial x} + \mu_2\frac{\partial u_2}{\partial x}\right)\Bigg|_{x=0} \frac{y\,\mathrm{d}y}{(y^2-\rho^2)^{1/2}} = 0 \tag{99}$$

for $\rho > 1$, so that on $x = 0$

$$\mu_1 \frac{\partial u_1}{\partial x} + \mu_2 \frac{\partial u_2}{\partial x} = \mu_1 \frac{\partial v_1(o, y)}{\partial y} + \mu_2 \frac{\partial v_2(o, y)}{\partial y} = 0, \qquad y > 1 \qquad [100]$$

or equivalently

$$\mu_1 v_1(o, y) + \mu_2 v_2(o, y) = A, \qquad y > 1 \qquad [101]$$

where A is a constant. Thus from [98] and [101] it follows that

$$v_1(o, y) = v_2(o, y) = \frac{A}{\mu_1 + \mu_2} = B, \qquad y > 1. \qquad [102]$$

Now for $\rho > 1$,

$$\begin{aligned} U_j(o, \rho) &= \int_0^1 \frac{v_j(o, y) y \, dy}{(\rho^2 - y^2)^{1/2}} + \int_1^\rho \frac{v_j(o, y) y \, dy}{(\rho^2 - y^2)^{1/2}} \\ &= \frac{2}{\pi} \int_0^1 \frac{y^2 \, dy}{(\rho^2 - y^2)^{1/2}} + B \int_1^\rho \frac{y \, dy}{(\rho^2 - y^2)^{1/2}} \\ &= \frac{\rho^2}{\pi} \sin^{-1} \frac{1}{\rho} + \left(B - \frac{1}{\pi}\right)(\rho^2 - 1)^{1/2} \end{aligned} \qquad [103]$$

and

$$\frac{1}{\rho} \frac{\partial U_j}{\partial \rho}\bigg|_{x=0} = \frac{2}{\pi} \sin^{-1} \frac{1}{\rho} + \left(B - \frac{2}{\pi}\right)(\rho^2 - 1)^{1/2}. \qquad [104]$$

Thus the velocity is singular at the rim of the disk unless $B = 2/\pi$, so that $v(x, y)$ can now be uniquely determined by the Green's function formula

$$v_j(x, y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{v_j(o, s) \, ds}{x^2 + (y - s)^2} \qquad [105]$$

where

$$\begin{aligned} v_j(o, s) &= \frac{2}{\pi}, \qquad |s| > 1 \\ &= \frac{2}{\pi}, \qquad s > 1 \\ &= -\frac{2}{\pi}, \qquad s > -1. \end{aligned} \qquad [106]$$

This is the same problem as [49] and the solution is expressed by [51] *viz.*

$$\begin{aligned} v_j(x, y) = \frac{2y}{\pi^2} \left\{ \tan^{-1}\left(\frac{1 - y}{x}\right) - \tan^{-1}\left(\frac{-1 - y}{x}\right) \right\} + \frac{x}{\pi^2} \log \left\{ \frac{x^2 + (1 - y)^2}{x^2 + (1 + y)^2} \right\} \\ + \frac{2}{\pi^2} \left\{ \tan^{-1}\left(\frac{-1 - y}{x}\right) - \tan^{-1}\left(\frac{1 - y}{x}\right) \right\}. \end{aligned} \qquad [107]$$

U_j can now be determined from the formula

$$\left.\begin{aligned} U_j(x, y) &= \int_0^\rho \frac{v_j(x, y) y \, dy}{(\rho^2 - y^2)^{1/2}} \qquad x > 0 \\ &= \int_0^\rho \frac{v_j(-x, y) y \, dy}{(\rho^2 - y^2)^{1/2}} \qquad x < 0. \end{aligned}\right\} \qquad [108]$$

It is observed that the velocity is independent of the ratio of viscosities and is the same as if the disk were moving broadside on through an infinite fluid of the same viscosity. The drag on the disk is finite and is given by

$$D = 8\bar{V}a(\mu_1 + \mu_2) \qquad [109]$$

where $\bar{V}$ is the velocity and a is the physical radius of the disk. The axial component of velocity on the interface is expressed by

$$u_x^{(j)} = -\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\rho}\right) \qquad [110]$$

which decays like ρ^{-1} as $\rho \to \infty$. In this model the disk carries the interface with the disk as it moves through the fluid except the interface is not disturbed at infinity.

4. DISK MOVING ACROSS AN INTERFACE AT REST

In this model the disk is moving broadside across an interface $x = 0$ in which there is zero normal velocity. It is evident there is a discontinuity in the velocity at the rim of the disk and the drag on the disk is infinite. However, the velocity field is derived since it is possible to find a field which converges analytically even though the drag is logarithmically infinite.

The fluid velocity field is again axisymmetric and may be written as

$$\mathbf{q}_j = \operatorname{curl}\left\{\frac{-\psi_j}{\rho}\,\hat{\boldsymbol{\phi}}\right\} \qquad j = 1, 2 \qquad [111]$$

where the stream function ψ_j satisfies the Stokes equation

$$L^2_{-1}(\psi_j) = 0. \qquad [112]$$

The velocity components are

$$u_x^{(j)} = -\frac{1}{\rho}\frac{\partial \psi_j}{\partial \rho}, \qquad u_\rho^{(j)} = \frac{1}{\rho}\frac{\partial \psi_j}{\partial x}. \qquad [113]$$

If the disk is moving broadside on with unit speed the boundary conditions on the disk are:

$$u_x^{(j)} = -1, \qquad u_\rho^{(j)} = 0 \qquad \text{at } x = 0, 0 \le \rho \le 1. \qquad [114]$$

On the interface the normal velocity is zero, that is

$$u_x^{(j)} = 0, \qquad \text{at } x = 0, \rho > 1. \qquad [115]$$

Also, the tangential component of velocity is continuous on the interface and this requires

$$u_\rho^{(1)} = u_\rho^{(2)} \qquad \text{at } x = 0, \rho > 1. \qquad [116]$$

The tangential component of stress is continuous at the interface, that is

$$p_{\rho x}^{(1)} = p_{\rho x}^{(2)} \qquad \text{at } x = 0, \rho > 1. \qquad [117]$$

The condition of continuous normal stress cannot in general be satisfied and it is implicitly assumed that the interfacial tension is sufficiently high to preclude large deformations of the interface.

In addition the velocity vanishes at infinity. This requires

$$(u_x^{(j)}, u_\rho^{(j)}) \to 0 \qquad \text{as} \qquad x^2+\rho^2\to\infty. \tag{118}$$

In terms of the stream function ψ_j the boundary conditions are:

(i)
$$\psi_j(o,\rho)=\frac{1}{2}\rho^2, \qquad \text{at } x=0,\ 0\le\rho\le 1 \tag{119}$$

(ii)
$$\frac{\partial\psi_j}{\partial x}(o,\rho)=0 \qquad \text{at } x=0,\ 0\le\rho\le 1 \tag{120}$$

(iii)
$$\psi_j=A \qquad \text{at } x=0,\ \rho>1 \tag{121}$$

(iv)
$$\frac{1}{\rho}\frac{\partial\psi_1}{\partial x}=\frac{1}{\rho}\frac{\partial\psi_2}{\partial x} \qquad \text{at } x=0,\ \rho>1 \tag{122}$$

(v)
$$\mu_1\left(\frac{\partial^2\psi_1}{\partial x^2}-\frac{\partial^2\psi_1}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi_1}{\partial\rho}\right)=\mu_2\left(\frac{\partial^2\psi_2}{\partial x^2}-\frac{\partial^2\psi_2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi_2}{\partial\rho}\right) \tag{123}$$

(vi)
$$\left(\frac{1}{\rho}\frac{\partial\psi_j}{\partial\rho},\frac{1}{\rho}\frac{\partial\psi_j}{\partial x}\right)\to 0 \qquad \text{as} \qquad x^2+\rho^2\to\infty. \tag{124}$$

An appropriate representation for the stream function ψ_j is expressed by

$$\psi_j=U_j-x\left(\frac{\partial U_j}{\partial x}-V_j\right) \tag{125}$$

where U_j, V_j again satisfy [78]. The problems for U_j and V_j are decoupled and may be stated as follows:

$$\left.\begin{aligned} U_j&=\frac{1}{2}\rho^2 \qquad \text{at } x=0,\ 0\le\rho\le 1\\ &=A \qquad \text{at } x=0,\ \rho>1\\ \left(\frac{1}{\rho}\frac{\partial U_j}{\partial\rho},\frac{1}{\rho}\frac{\partial U_j}{\partial x}\right)&\to 0 \qquad \text{as} \qquad x^2+\rho^2\to\infty\end{aligned}\right. \tag{126}$$

and for V_j:

$$\left.\begin{aligned} &V_1=V_2=0 \qquad \text{at } x=0,\ 0\le\rho\le 1\\ &V_1=V_2 \qquad \text{at } x=0,\ \rho>1\\ &\mu_1\frac{\partial V_1}{\partial x}=\mu_2\frac{\partial V_2}{\partial x} \qquad \text{at } x=0,\ \rho>1\\ &V_j\to 0 \qquad \text{as} \qquad x^2+\rho^2\to\infty.\end{aligned}\right\} \tag{127}$$

The solution of this latter mixed boundary value problem is identically zero. The solution of the former boundary value problem for V_j can be found using a Hankel transform. A suitable representation for the solution of [126] in $x\ge 0$ is given by

$$V_j(x,\rho)=A+\int_0^\infty f(k)e^{-kx}\rho J_1(k\rho)k\,dk \tag{128}$$

where $J_1(k\rho)$ is the Bessel function of the first kind and first order and $f(k)$ is to be determined on the plane $x=0$

$$\int_0^\infty f(k)\rho J_1(k\rho)k\,\mathrm{d}k = \frac{1}{2}\rho^2 - A, \qquad 0 \le \rho \le 1$$
$$= 0, \qquad \rho > 1. \qquad [129]$$

The inverse formula for the Hankel transform yields

$$f(k) = \int_0^1 \left(\frac{1}{2}\rho^2 - A\right) J_1(k\rho)\,\mathrm{d}\rho$$
$$= \frac{J_1(k)}{k^2} + \left(A - \frac{1}{2}\right)\frac{J_0(k)}{k}. \qquad [130]$$

Thus, $V_j(x,\rho)$ is expressed by

$$V_j(x,\rho) = A + \int_0^\infty \left(\frac{J_1(k)}{k} + \left(A - \frac{1}{2}\right)J_0(k)\right) \mathrm{e}^{-kx}\rho J_1(k\rho)\,\mathrm{d}k \qquad [131]$$

where the constant A to the present is arbitrary. The singularity in the stresses and vorticity at $\rho = 1$ is minimized if $A = 1/2$ so that

$$V_j(x,\rho) = \frac{1}{2} + \rho\int_0^\infty \frac{J_1(k)J_1(k\rho)}{k}\mathrm{e}^{-kx}\,\mathrm{d}k, \qquad x > 0. \qquad [132]$$

It is readily checked that [132] gives rise to an infinite force on the disk as may be expected. It is also noted that since $V_j = 0$, there is no tangential velocity on the interface. Also the velocity field is independent of viscosity so that the motion is the same as if the fluid were of one viscosity in which a disk is moving through a circular gap in a rigid plane wall. There is no tangential stress on the interface.

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